— Problems —

1. **Riemann-Lebesgue lemma.** Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic piecewise continuous function. Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. We want to show that

$$\lim_{n\to+\infty} \int_a^b f(t)e^{int}dt = 0.$$

   (a) Show that

$$\lim_{n\to+\infty} \int_a^b f(t)\varphi(nt)dt = \frac{1}{2\pi}\left(\int_0^{2\pi} \varphi(t)dt\right)\left(\int_a^b f(t)dt\right).$$

   Hints: Denote $K = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)dt$. *We want to show that*

$$\lim_{n\to+\infty} \int_a^b f(t)\Big(\varphi(nt) - K\Big)dt = 0, \textit{ i.e.,}$$

   $\lim_{n\to+\infty} \int_a^b f(t)\psi(nt)dt = 0$ *with* $\psi = \varphi - K$. *Note that* $\int_0^{2\pi} \psi(t)dt = 0$. *Prove the result in three steps: first when* f *is a characteristic function* $\chi_{[\alpha,\beta]}$ *with* $[\alpha, \beta] \subset [a, b]$, *then when* f *is a step function, then for a general* f, *which can be approximated by a step function.*

   (b) Show that $\int_0^{2\pi} e^{it}dt = 0$ and conclude.

2. **Fejér's theorem.** Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and 2π-periodic. For all $n \geq 0$, define the functions

$$S_n = \sum_{k=-n}^{n} c_k(f)e_k, \quad C_n = \frac{S_0 + \cdots + S_n}{n+1}$$

and

$$\tilde{S}_n = \sum_{k=-n}^{n} e_k, \quad \tilde{C}_n = \frac{\tilde{S}_0 + \cdots + \tilde{S}_n}{n+1}.$$

   (a) Check that for all n, $\frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{C}_n(t)dt = 1$, and show that $\forall\alpha \in (0, \pi)$, the sequence of functions $(\tilde{C}_n)$ converges uniformly to 0 on $[-\pi, \pi] \setminus [-\alpha, \alpha]$.

   (b) Deduce from this that the sequence of functions (C_n) converges uniformly to f on $\mathbb{R}$. In particular, f can be uniformly approximated by trigonometric polynomials.